

# On Weighted Measure of Inaccuracy for Doubly Truncated Random Variables

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## Abstract

Recently, authors have studied weighted version of Kerridge inaccuracy measure for truncated distributions. In the present communication we introduce the notion of weighted interval inaccuracy measure for two-sided truncated random variables. In reliability theory and survival analysis, this measure may help to study the various characteristics of a system/component when it fails between two time points. Various aspects of weighted interval inaccuracy measure have been discussed and some characterization results have been provided. This new measure is a generalization of recent dynamic weighted inaccuracy measure.

**Key Words and Phrases:** Entropy, weighted inaccuracy measure, proportional (reversed) hazard model.

**AMS 2010 Classifications:** Primary 94A17; Secondary 62N05, 62E10.

## 1 Introduction

The idea of information theoretic entropy was introduced by Shannon (1948) and Weiner (1949). Shannon was the one who formally introduced entropy, known as *Shannon's entropy* or *Shannon's information measure*, into information theory, and characterized the properties of information sources and of communication channels to analyze the outputs of these sources.

Let us consider an absolutely continuous nonnegative random variable  $X$  with probability density function  $f$ , distribution function  $F$  and survival function  $\overline{F} \equiv 1 - F$ . Then the Shannon's information measure or the differential entropy of  $X$  is given by

$$H_X = - \int_0^{\infty} f(x) \ln f(x) dx, \quad (1.1)$$

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which measures the expected uncertainty contained in  $f(\cdot)$  about the predictability of an outcome of  $X$ .

Since the pioneering contributions by Shannon and Weiner, numerous efforts have been made to enrich and extend the underlying information theory. One important development in this direction is inaccuracy measure due to Kerridge (1961) which can be thought of as a generalization of Shannon's entropy. It has been extensively used as a useful tool for measurement of error in experimental results. Suppose that an experimenter states the probabilities of the various possible outcomes of an experiment. His statement can lack precision in two ways: he may not have enough information and so his statement is vague, or some of the information he has may be incorrect. All statistical inference related problems are concerned with making statements which may be inaccurate in either or both of these ways. Kerridge (1961) proposed the *inaccuracy measure* that can take accounts for these two types of errors. Suppose that the experimenter asserts that the probability of the  $i^{th}$  eventuality is  $q_i$  when the true probability is  $p_i$ . Then the inaccuracy of the observer can be measured by

$$I(P, Q) = - \sum_{i=1}^n p_i \ln q_i,$$

where  $P = (p_1, p_2, \dots, p_n)$  and  $Q = (q_1, q_2, \dots, q_n)$  are two discrete probability distributions such that  $p_i \geq 0$ ,  $q_i \geq 0$  and  $\sum_{i=1}^n p_i = 1 = \sum_{i=1}^n q_i$ .

Nath (1968) extended Kerridge's inaccuracy measure to the case of continuous situation and discussed some properties. If  $F(x)$  is the actual distribution corresponding to the observations and  $G(x)$  is the distribution assigned by the experimenter and  $f$ ,  $g$  are the corresponding density functions, then the inaccuracy measure is defined as

$$H_{X,Y} = - \int_0^\infty f(x) \ln g(x) dx. \quad (1.2)$$

It has applications in statistical inference and coding theory. When  $g(x) = f(x)$ , then (1.2) becomes (1.1), the Shannon's entropy. The definition of inaccuracy measure was also extended to truncated situation, see, Nair and Gupta (2007), Taneja et al. (2009) and Kumar et al. (2011) for further details.

It is well-known that Shannon entropy is a shift independent measure. However, in certain applied contexts, such as reliability or mathematical neurobiology, it is desirable to deal with shift-dependent information measures. Indeed, knowing that a device fails to operate, or a neuron to release spikes in a given time-interval, yields relevantly different information from the case when such an event occurs in a different equally wide interval. In some cases we are thus led to resort to a shift-dependent information measure that, for instance, assigns different measures to such distributions. Also, there exist many fields dealing with random experiment whose elementary events are characterized both by their objective probabilities and by some qualitative (objective or subjective) weights attached to elementary events and which may or may not be dependent on the objective probabilities.

In analogy with Belis and Guiaşu (1968), Di Crescenzo and Longobardi (2006) considered the notion of weighted entropy

$$H_X^w = - \int_0^\infty x f(x) \ln f(x) dx. \quad (1.3)$$

As pointed out by Belis and Guiaşu (1968) that the occurrence of an event removes a double uncertainty: the quantitative one, related to the probability with which it occurs, and the qualitative one, related to its utility for the attainment of the goal or to its significance with respect to a given qualitative characteristic. The factor  $x$ , in the integral on the right-hand-side of (1.3), may be viewed as a weight linearly emphasizing the occurrence of the event  $\{X = x\}$ . This yields a length biased shift-dependent information measure assigning greater importance to larger values of  $X$ . The use of weighted entropy (1.3) is also motivated by the need, arising in various communication and transmission problems, of expressing the usefulness of events by means of an information measure.

In agreement with Taneja and Tuteja (1986), here we consider the weighted inaccuracy measure

$$H_{X,Y}^w = - \int_0^\infty x f(x) \ln g(x) dx, \quad (1.4)$$

which is a quantitative-qualitative measure of inaccuracy associated with the statement of an experimenter. When  $g(x) = f(x)$ , then (1.4) becomes (1.3), the weighted entropy. For more properties of quantitative-qualitative measure of inaccuracy one may refer to Prakash and Taneja (1986) and Bhatia and Taneja (1991), among others. The following example illustrates the importance of qualitative characteristic of information as reflected in the definition of weighted inaccuracy measure.

**Example 1.1** *Let  $X_1$  and  $Y_1$  denote random lifetimes of two components with probability density functions  $f_1(x) = x/2$ ,  $x \in (0, 2)$  and  $g_1(x) = (2 - x)/2$ ,  $x \in (0, 2)$  respectively. By simple calculations, we have  $H_{X_1, Y_1} = H_{Y_1, X_1} = 3/2$ . But,*

$$H_{X_1, Y_1}^w = \frac{22}{9} \text{ and } H_{Y_1, X_1}^w = \frac{5}{9}.$$

*Therefore, the inaccuracy measure of the observer for the observations  $X_1$  (resp.  $Y_1$ ) taking  $Y_1$  (resp.  $X_1$ ) as corresponding assigned outcomes by the experimenter are identical. Instead,  $H_{X_1, Y_1}^w > H_{Y_1, X_1}^w$ , i.e., weighted inaccuracy of the observer for  $(X_1, Y_1)$  is higher than that for  $(Y_1, X_1)$ . As a matter of fact, the inaccuracies measured from a quantitative point of view, neglecting the qualitative side, are identical. To distinguish them, we must take into account the qualitative characteristic as given in (1.4).  $\square$*

Analogous to weighted residual and past entropies Kumar et al. (2010) and Kumar and Taneja (2012) introduced the notion of weighted residual inaccuracy measure given by

$$H_{X,Y}^w(t) = - \int_t^\infty x \frac{f(x)}{F(t)} \ln \left( \frac{g(x)}{G(t)} \right) dx \quad (1.5)$$

and weighted past inaccuracy measure given by

$$\overline{H}_{X,Y}^w(t) = - \int_0^t x \frac{f(x)}{F(t)} \ln \left( \frac{g(x)}{G(t)} \right) dx, \quad (1.6)$$

and studied their properties in analogy with weighted residual entropy and weighted past entropy, respectively. For  $t = 0$ , (1.5) reduces to (1.4) and for  $t = \infty$ , (1.6) reduces to (1.4). Various aspects of (1.5) and (1.6) have been discussed in Kundu (2014).

The rest of the paper is arranged as follows. In Section 2 we introduce the concept of weighted interval inaccuracy measure for doubly truncated random variables. We obtain upper and lower bounds for weighted interval inaccuracy measure. In Section 3 we provide characterizations of quite a few useful continuous distributions based on this newly introduced measure including its uniqueness property. The effect of monotone transformations on the weighted interval inaccuracy measure has been discussed in Section 4.

## 2 Weighted interval inaccuracy measure

In the study of income distribution, the inequality is computed not only for income greater/smaller than a fixed value but also for income between two values. For example, in many practical situations, it is of interest to study the inequality of a population eliminating high (richest population) and low (poorest population) values, and therefore doubly truncated populations are considered. In reliability theory and survival analysis, often individuals whose event time lies within a certain time interval are only observed and one has information about the lifetime between two time points. Thus, an individual whose event time is not in this interval is not observed and therefore information on the subjects outside this interval is not available to the investigator. Accordingly, Kotlarski (1972) studied the conditional expectation for the doubly truncated random variables. Later, Navarro and Ruiz (1996) generalized the failure rate and the conditional expectation to the doubly truncated random variables. For various related results one may refer to Ruiz and Navarro (1996), Betensky and Martin (2003), Sankaran and Sunoj (2004) among others. Recently, Sunoj et al. (2009) and Misagh and Yari (2010, 2012) studied the measure of uncertainty and conditional measure for doubly truncated random variables and obtained some characterization results. Furthermore, Misagh and Yari (2011) explored the use of weighted information measures for doubly truncated random variables. Motivated by this, we introduce the notion of weighted interval inaccuracy measure for doubly truncated random variables.

Let us consider two nonnegative absolutely continuous doubly truncated random variables  $(X|t_1 \leq X \leq t_2)$  and  $(Y|t_1 \leq Y \leq t_2)$  where  $(t_1, t_2) \in D = \{(u, v) \in \mathbb{R}_+^2 : F(u) < F(v) \text{ and } G(u) < G(v)\}$ . Then the interval inaccuracy measure of  $X$  and  $Y$  at interval  $(t_1, t_2)$

is given by

$$H_{X,Y}(t_1, t_2) = - \int_{t_1}^{t_2} \frac{f(x)}{F(t_2) - F(t_1)} \ln \frac{g(x)}{G(t_2) - G(t_1)} dx. \quad (2.7)$$

When  $g(x) = f(x)$ , we obtain measure of uncertainty for doubly truncated random variable as given in (2.6) and (2.7) of Sunoj et al. (2009). Various aspects of interval inaccuracy measure have been discussed in Kundu and Nanda (2014). To construct a shift-dependent dynamic measure of inaccuracy, we use (2.7) and define weighted interval inaccuracy measure for two-sided truncated random variables.

**Definition 2.1** *The weighted interval inaccuracy measure of  $X$  and  $Y$  at interval  $(t_1, t_2)$  is given by*

$$H_{X,Y}^w(t_1, t_2) = - \int_{t_1}^{t_2} x \frac{f(x)}{F(t_2) - F(t_1)} \ln \frac{g(x)}{G(t_2) - G(t_1)} dx. \quad (2.8)$$

**Remark 2.1** *Clearly,  $H_{X,Y}^w(0, t) = \overline{H}_{X,Y}^w(t)$ ,  $H_{X,Y}^w(t, \infty) = H_{X,Y}^w(t)$  and  $H_{X,Y}^w(0, \infty) = H_{X,Y}^w$  as given in (1.6), (1.5) and (1.4) respectively.  $\square$*

The following example clarifies the effectiveness of the weighted interval inaccuracy measure.

**Example 2.1** *Let  $X_1, Y_1$  be the random lifetimes as given in Example 1.1. Also let  $X_2, Y_2$  denote random lifetimes of two components with probability density functions  $f_2(x) = 2x$ ,  $x \in (0, 1)$  and  $g_2(x) = 2(1 - x)$ ,  $x \in (0, 1)$  respectively. Since  $X_1, Y_1$  and  $X_2, Y_2$  belong to different domains, the use of weighted inaccuracy measure (1.4) to compare them informatively is not interpretable. The weighted interval inaccuracy measure in the interval  $(0.2, 0.8)$  are  $H_{X_1, Y_1}^w(0.2, 0.8) = -0.1143$  and  $H_{X_2, Y_2}^w(0.2, 0.8) = -0.2416$ . Hence, the weighted interval inaccuracy measure between  $X_1, Y_1$  is greater than of it between  $X_2, Y_2$  in the interval  $(0.2, 0.8)$ .  $\square$*

An alternative way of writing (2.8) is as follows:

$$H_{X,Y}^w(t_1, t_2) = - \frac{1}{F(t_2) - F(t_1)} \int_{t_1}^{t_2} x f(x) \ln g(x) dx + \frac{\ln\{G(t_2) - G(t_1)\}}{F(t_2) - F(t_1)} \int_{t_1}^{t_2} x f(x) dx,$$

where the second integral on the right hand side is equal to

$$t_2 F(t_2) - t_1 F(t_1) - \int_{t_1}^{t_2} F(x) dx, \quad \text{or} \quad t_1 \overline{F}(t_1) - t_2 \overline{F}(t_2) + \int_{t_1}^{t_2} \overline{F}(x) dx.$$

The weighted interval inaccuracy measure can also be written as

$$\begin{aligned} H_{X,Y}^w(t_1, t_2) &= - \int_{t_1}^{t_2} \int_0^x \frac{f(x)}{F(t_2) - F(t_1)} \ln \frac{g(x)}{G(t_2) - G(t_1)} dy dx \\ &= t_1 H_{X,Y}(t_1, t_2) + \int_{t_1}^{t_2} H_{X,Y}(x, t_2) dx. \end{aligned} \quad (2.9)$$

Furthermore,

$$H_{X,Y}^w(t_1, t_2) = t_2 H_{X,Y}(t_1, t_2) - \int_{t_1}^{t_2} H_{X,Y}(t_1, y) dy, \quad (2.10)$$

where  $H_{X,Y}(t_1, t_2)$  is the interval inaccuracy measure given in (2.7). Differentiating (2.9) and (2.10) with respect to  $t_1$  and  $t_2$ , respectively, we obtain

$$\frac{\partial}{\partial t_1} H_{X,Y}^w(t_1, t_2) = t_1 \frac{\partial}{\partial t_1} H_{X,Y}(t_1, t_2) \quad \text{and} \quad \frac{\partial}{\partial t_2} H_{X,Y}^w(t_1, t_2) = t_2 \frac{\partial}{\partial t_2} H_{X,Y}(t_1, t_2).$$

**Remark 2.2** *Weighted interval inaccuracy measure is increasing (decreasing) in  $t_1$  if and only if the interval inaccuracy measure is increasing (decreasing) in  $t_1$ . The result also holds for  $t_2$ .*  $\square$

We decompose the weighted Kerridge inaccuracy measure in terms of weighted residual, past and interval inaccuracy measures on using the similar approach to that of Misagh and Yari (2011).

**Remark 2.3** *Let  $X$  and  $Y$  be two absolutely continuous nonnegative random variables with  $E(X) < \infty$ . Then, for all  $0 < t_1 < t_2 < \infty$ , the weighted Kerridge inaccuracy measure can be decomposed as*

$$\begin{aligned} H_{X,Y}^w &= F(t_1) \overline{H}_{X,Y}^w(t_1) + [F(t_2) - F(t_1)] H_{X,Y}^w(t_1, t_2) + \overline{F}(t_2) H_{X,Y}^w(t_2) \\ &\quad - E(X) \left[ F^*(t_1) \ln G(t_1) + \{F^*(t_2) - F^*(t_1)\} \ln \{G(t_2) - G(t_1)\} + \overline{F}^*(t_2) \ln \overline{G}(t_2) \right], \end{aligned}$$

which can be interpreted as follows. The weighted inaccuracy measure can be decomposed into four parts: (i) the weighted inaccuracy measure for random variables truncated above  $t_1$ , (ii) the weighted inaccuracy measure in the interval  $(t_1, t_2)$  given that the item has failed after  $t_1$  but before  $t_2$ , (iii) the weighted inaccuracy measure for random variables truncated below  $t_2$  and (iv) the pseudo inaccuracy for trivalent random variables which determines whether the item has failed before  $t_1$  or in between  $t_1$  and  $t_2$  or after  $t_2$ .

When  $t_1 = t_2 = t$ , then the above can be written as

$$H_{X,Y}^w = F(t) \overline{H}_{X,Y}^w(t) + \overline{F}(t) H_{X,Y}^w(t) - E(X) \left[ F^*(t) \ln G(t) + \overline{F}^*(t) \ln \overline{G}(t) \right],$$

a result obtained by Kumar and Taneja (2012).  $\square$

In virtue of Remark 2.2, below we obtain the bounds for the interval inaccuracy measure based on the monotonic behavior of the weighted interval inaccuracy measure. We first give definitions of general failure rate (GFR), general conditional mean (GCM) and geometric vitality function of a random variable  $X$  truncated at two points  $t_1$  and  $t_2$  where  $(t_1, t_2) \in D$ . For details one may refer to Navarro and Ruiz (1996), Nair and Rajesh (2000) and Sunoj et al. (2009).

**Definition 2.2** The GFR functions of a doubly truncated random variable  $(X|t_1 < X < t_2)$  are given by  $h_1^X(t_1, t_2) = \frac{f(t_1)}{F(t_2) - F(t_1)}$  and  $h_2^X(t_1, t_2) = \frac{f(t_2)}{F(t_2) - F(t_1)}$ . Similarly  $h_1^Y(t_1, t_2)$  and  $h_2^Y(t_1, t_2)$  are defined for the random variable  $(Y|t_1 < Y < t_2)$ .  $\square$

**Definition 2.3** The GCM of a doubly truncated random variable  $(X|t_1 < X < t_2)$  is defined by

$$m_X(t_1, t_2) = E(X|t_1 < X < t_2) = \frac{1}{F(t_2) - F(t_1)} \int_{t_1}^{t_2} x f(x) dx.$$

**Definition 2.4** The geometric vitality function for doubly truncated random variable  $(X|t_1 < X < t_2)$  is given by

$$\mathcal{G}_X(t_1, t_2) = E(\ln X|t_1 < X < t_2),$$

which gives the geometric mean life of  $X$  truncated at two points  $t_1$  and  $t_2$ , provided  $E(\ln X)$  is finite. The corresponding weighted version of it is given by  $\mathcal{G}_X^w(t_1, t_2) = E(X \ln X|t_1 < X < t_2)$ .  $\square$

When  $H_{X,Y}^w(t_1, t_2)$  is increasing in each of the arguments keeping the other fixed, then on differentiating (2.8) with respect to  $t_1$  and  $t_2$ , we get

$$\frac{h_1^Y(t_1, t_2)}{h_1^X(t_1, t_2)} - \ln h_1^Y(t_1, t_2) \leq H_{X,Y}(t_1, t_2) \leq \frac{h_2^Y(t_1, t_2)}{h_2^X(t_1, t_2)} - \ln h_2^Y(t_1, t_2).$$

The following proposition gives bounds for the weighted interval inaccuracy measure. The proof follows from (2.8) and hence omitted.

**Proposition 2.1** If  $g(x)$  is decreasing in  $x > 0$ , then

$$-m_X(t_1, t_2) \ln h_1^Y(t_1, t_2) \leq H_{X,Y}^w(t_1, t_2) \leq -m_X(t_1, t_2) \ln h_2^Y(t_1, t_2).$$

For increasing  $g(x)$  the above inequalities are reversed.  $\square$

In the following two theorems we provide upper and lower bounds for the weighted interval inaccuracy measure based on monotonic behavior of the GFR functions of  $Y$ .

**Theorem 2.1** For fixed  $t_2$ ,

(i) if  $h_1^Y(t_1, t_2)$  is decreasing in  $t_1$  then  $H_{X,Y}^w(t_1, t_2) \geq -m_X(t_1, t_2) \ln h_1^Y(t_1, t_2)$ ,

and (ii) increasing  $h_1^Y(t_1, t_2)$  in  $t_1$  implies

$$H_{X,Y}^w(t_1, t_2) \leq -m_X(t_1, t_2) \ln h_1^Y(t_1, t_2) - \int_{t_1}^{t_2} \frac{x f(x)}{F(t_2) - F(t_1)} \ln \frac{G(t_2) - G(x)}{G(t_2) - G(t_1)} dx.$$

Proof: Note that (2.8) can be written as

$$H_{X,Y}^w(t_1, t_2) = - \int_{t_1}^{t_2} \frac{x f(x) \ln h_1^Y(x, t_2)}{F(t_2) - F(t_1)} dx - \int_{t_1}^{t_2} \frac{x f(x)}{F(t_2) - F(t_1)} \ln \frac{G(t_2) - G(x)}{G(t_2) - G(t_1)} dx. \quad (2.11)$$

(i) For  $t_1 < x$ ,  $\ln \frac{G(t_2) - G(x)}{G(t_2) - G(t_1)} \leq 0$  and  $\ln h_1^Y(x, t_2) \leq \ln h_1^Y(t_1, t_2)$  if  $h_1^Y(t_1, t_2)$  is decreasing in  $t_1$ . Then, from (2.11), we obtain

$$\begin{aligned} H_{X,Y}^w(t_1, t_2) &\geq - \int_{t_1}^{t_2} \frac{x f(x)}{F(t_2) - F(t_1)} \ln h_1^Y(x, t_2) dx \\ &\geq -m_X(t_1, t_2) \ln h_1^Y(t_1, t_2). \end{aligned}$$

(ii) The second part follows easily from (2.11) on using the fact that  $\ln h_1^Y(x, t_2) \geq \ln h_1^Y(t_1, t_2)$  for  $t_1 < x$ .  $\square$

**Remark 2.4** In the above theorem if we take  $t_2 = \infty$ , then we get the lower (resp. upper) bound for the weighted residual inaccuracy measure as obtained by Kumar et al. (2010) (resp. Kundu, 2014).  $\square$

**Example 2.2** Let  $X$  be a nonnegative random variable with probability density function

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 0, & \text{otherwise} \end{cases} \quad (2.12)$$

and  $Y$  is uniformly distributed over  $(0, a)$ . Then  $m_X(t_1, t_2) = \frac{2(t_1^2 + t_1 t_2 + t_2^2)}{3(t_1 + t_2)}$ ,  $h_1^Y(t_1, t_2) = \frac{1}{(t_2 - t_1)}$  and  $H_{X,Y}^w(t_1, t_2) = \frac{2(t_1^2 + t_1 t_2 + t_2^2) \ln(t_2 - t_1)}{3(t_1 + t_2)}$ . Note that right hand side of part (ii) is  $\geq \frac{2(t_1^2 + t_1 t_2 + t_2^2) \ln(t_2 - t_1)}{3(t_1 + t_2)}$ . It is easily seen that part (ii) of the above theorem is fulfilled. For part (i), let  $X$  be uniformly distributed over  $[\alpha, \beta]$  and let  $Y$  follow Pareto-I distribution given by

$$G(t) = 1 - \frac{\alpha}{t}, \quad t > \alpha (> 0). \quad (2.13)$$

Then  $m_X(t_1, t_2) = \frac{(t_1 + t_2)}{2}$ ,  $\alpha < t_1 < t_2 < \beta$  and  $h_1^Y(t_1, t_2) = \frac{t_2}{t_1(t_2 - t_1)}$ , which is decreasing in  $t_1$ , for fixed  $t_2 > 2t_1$ . Now

$$\begin{aligned} H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_1^Y(t_1, t_2) &= 2 \left[ \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} x \ln x dx - \frac{(t_1 + t_2)}{2} \ln t_1 \right] \\ &\geq 0, \end{aligned}$$

and equality holds for  $t_1 \rightarrow t_2$ . Hence part (i) is also fulfilled.  $\square$

The proof of the following theorem is analogous to Theorem 2.1 but for completeness we give a brief outline of the proof.

**Theorem 2.2** For fixed  $t_1$ , if  $h_2^Y(t_1, t_2)$  is decreasing in  $t_2$  then

$$H_{X,Y}^w(t_1, t_2) \leq -m_X(t_1, t_2) \ln h_2^Y(t_1, t_2) - \int_{t_1}^{t_2} \frac{xf(x)}{F(t_2) - F(t_1)} \ln \frac{G(x) - G(t_1)}{G(t_2) - G(t_1)} dx.$$

Proof: We write (2.8) as

$$H_{X,Y}^w(t_1, t_2) = - \int_{t_1}^{t_2} \frac{xf(x) \ln h_2^Y(t_1, x)}{F(t_2) - F(t_1)} dx - \int_{t_1}^{t_2} \frac{xf(x)}{F(t_2) - F(t_1)} \ln \frac{G(x) - G(t_1)}{G(t_2) - G(t_1)} dx. \quad (2.14)$$

Hence the result follows from (2.14) on using the fact that, for  $x < t_2$ ,  $\ln h_2^Y(t_1, x) \geq \ln h_2^Y(t_1, t_2)$  when  $h_2^Y(t_1, t_2)$  is decreasing in  $t_2$ .  $\square$

**Remark 2.5** If in the above theorem we take  $t_1 = 0$ , then we get

$$\overline{H}_{X,Y}^w(t_2) \leq -\tau_F(t_2) [\ln \phi_G(t_2) + 1] - \frac{G(t_2)}{F(t_2)} \int_0^{t_2} \frac{xf(x)}{G(x)} dx,$$

an upper bound to the weighted past inaccuracy measure as obtained in Theorem 4.2 of Kumar and Taneja (2012).



**Example 2.3** Let  $X$  be a nonnegative random variable with probability density function as given in (2.12) and let  $Y$  be uniformly distributed over  $(0, a)$ . Since  $h_1^Y(t_1, t_2) = h_2^Y(t_1, t_2) = \frac{1}{(t_2 - t_1)}$ , on using the same argument as in Example 2.2, it can easily be shown that the condition of the above theorem is fulfilled.  $\square$

**Remark 2.6** It is not difficult to see from (2.14) that, for fixed  $t_1$ , if  $h_2^Y(t_1, t_2)$  is increasing in  $t_2$  then  $H_{X,Y}^w(t_1, t_2) \geq -m_X(t_1, t_2) \ln h_2^Y(t_1, t_2)$ . But it also can be shown that for a random variable with support  $[0, \infty)$ ,  $h_2^Y(t_1, t_2)$  may not be increasing in  $t_2$ . This condition can be achieved if either the support of the random variable is  $(-\infty, b]$  with  $b > 0$  or  $[0, b]$  with  $b < \infty$ .

### 3 Characterizations based on weighted interval inaccuracy measure

In the literature, the problem of characterizing probability distributions has been investigated by many researchers. The standard practice in modeling statistical data is either to derive the appropriate model based on the physical properties of the system or to choose a flexible family of distributions and then find a member of the family that is appropriate to the data. In both the situations it would be helpful if we find characterization theorems that explain the distribution. In fact, characterization approach is very appealing to both theoreticians and applied workers. In this section we show that weighted interval inaccuracy measure can uniquely determine the distribution function. We also provide characterizations of quite a few useful continuous distributions in terms of weighted interval inaccuracy measure.

First we define the proportional hazard rate model (PHRM) and proportional reversed hazard rate model (PRHRM). Let  $X$  and  $Y$  be two random variables with hazard rate functions  $h_F(\cdot)$ ,  $h_G(\cdot)$  and reversed hazard rate functions  $\phi_F(\cdot)$ ,  $\phi_G(\cdot)$ , respectively. Then  $X$  and  $Y$  are said to satisfy the PHRM (cf. Cox, 1959), if there exists  $\theta > 0$  such that  $h_G(t) = \theta h_F(t)$ , or equivalently,  $\overline{G}(t) = [\overline{F}(t)]^\theta$ , for some  $\theta$ . This model has been widely used in analyzing survival data; see, for instance, Cox (1972), Ebrahimi and Kirmani (1996), Gupta and Han (2001) and Nair and Gupta (2007) among others. Similarly,  $X$  and  $Y$  are said to satisfy PRHRM proposed by Gupta et al. (1998) in contrast to the celebrated PHRM with proportionality constant  $\theta > 0$ , if  $\phi_G(t) = \theta \phi_F(t)$ . Or, equivalently,  $G(t) = [F(t)]^\theta$ , for some  $\theta$ . This model is flexible enough to accommodate both monotonic as well as non-monotonic failure rates even though the baseline failure rate is monotonic. See Sengupta et al. (1999), Di Crescenzo (2000) or Gupta and Gupta (2007) for some results on this model.

The general characterization problem is to obtain when the weighted interval inaccuracy measure uniquely determines the distribution function. We consider the following characterization result. For characterization of a distribution by using its GFR functions one may refer to Navarro and Ruiz (1996).

**Theorem 3.1** For two absolutely continuous nonnegative random variables  $X$  and  $Y$ , when

$H_{X,Y}^w(t_1, t_2)$  is increasing in  $t_1$  (for fixed  $t_2$ ) and decreasing in  $t_2$  (for fixed  $t_1$ ) and  $h_i^Y(t_1, t_2) = \theta h_i^X(t_1, t_2)$ ,  $\theta > 0$ ,  $i = 1, 2$ , respectively, then  $H_{X,Y}^w(t_1, t_2)$  uniquely determines  $F(x)$ .

Proof: Differentiating (2.8) with respect to  $t_i$ ,  $i = 1, 2$ , we have

$$\frac{\partial}{\partial t_1} H_{X,Y}^w(t_1, t_2) = t_1 h_1^X(t_1, t_2) [H_{X,Y}(t_1, t_2) + \ln \theta - \theta + \ln h_1^X(t_1, t_2)]$$

$$\text{and, } \frac{\partial}{\partial t_2} H_{X,Y}^w(t_1, t_2) = -t_2 h_2^X(t_1, t_2) [H_{X,Y}(t_1, t_2) + \ln \theta - \theta + \ln h_2^X(t_1, t_2)].$$

Then for any fixed  $t_1$  and arbitrary  $t_2$ ,  $h_1^X(t_1, t_2)$  is a positive solution of the equation  $\eta(x_{t_2}) = 0$ , where

$$\eta(x_{t_2}) = t_1 x_{t_2} [H_{X,Y}(t_1, t_2) + \ln \theta - \theta + \ln x_{t_2}] - \frac{\partial}{\partial t_1} H_{X,Y}^w(t_1, t_2).$$

Similarly, for any fixed  $t_2$  and arbitrary  $t_1$ ,  $h_2^X(t_1, t_2)$  is a positive solution of the equation  $\zeta(y_{t_1}) = 0$ , where

$$\zeta(y_{t_1}) = t_2 y_{t_1} [H_{X,Y}(t_1, t_2) + \ln \theta - \theta + \ln y_{t_1}] + \frac{\partial}{\partial t_2} H_{X,Y}^w(t_1, t_2).$$

Differentiating  $\eta(x_{t_2})$  and  $\zeta(y_{t_1})$  with respect to  $x_{t_2}$  and  $y_{t_1}$ , respectively, we get  $\frac{\partial \eta(x_{t_2})}{\partial x_{t_2}} = t_1 [H_{X,Y}(t_1, t_2) + \ln \theta - \theta + 1 + \ln x_{t_2}]$  and  $\frac{\partial \zeta(y_{t_1})}{\partial y_{t_1}} = t_2 [H_{X,Y}(t_1, t_2) + \ln \theta - \theta + 1 + \ln y_{t_1}]$ . Furthermore, second order derivatives are  $\frac{\partial^2 \eta(x_{t_2})}{\partial x_{t_2}^2} = \frac{t_1}{x_{t_2}} > 0$  and  $\frac{\partial^2 \zeta(y_{t_1})}{\partial y_{t_1}^2} = \frac{t_2}{y_{t_1}} > 0$ . So, both the functions  $\eta(x_{t_2})$  and  $\zeta(y_{t_1})$  are minimized at  $x_{t_2} = \exp[\theta - \ln \theta - 1 - H_{X,Y}(t_1, t_2)] = y_{t_1}$ , respectively. Here  $\eta(0) = -\frac{\partial}{\partial t_1} H_{X,Y}^w(t_1, t_2) < 0$ , since we assume that  $H_{X,Y}^w(t_1, t_2)$  is increasing in  $t_1$ , and also, when  $x_{t_2} \rightarrow \infty$ ,  $\eta(x_{t_2}) \rightarrow \infty$ . Similarly  $\zeta(0) = \frac{\partial}{\partial t_2} H_{X,Y}^w(t_1, t_2) < 0$ , and  $\zeta(y_{t_1}) \rightarrow \infty$  as  $y_{t_1} \rightarrow \infty$ . Therefore, both the equations  $\eta(x_{t_2}) = 0$  and  $\zeta(y_{t_1}) = 0$  have unique positive solutions  $h_1^X(t_1, t_2)$  and  $h_2^X(t_1, t_2)$ , respectively. Hence the proof is completed on using the fact that GFR functions uniquely determine the distribution function (cf. Navarro and Ruiz, 1996).  $\square$

Now we provide characterization theorems for some continuous distributions using GFR, GCM, geometric vitality function and weighted interval inaccuracy measure under PHRM and PRHRM. Below we characterize uniform distribution. Recall that  $\frac{\partial h_1^Y(t_1, t_2)}{\partial t_2} = -h_1^Y(t_1, t_2) h_2^Y(t_1, t_2)$  and  $\frac{\partial h_1^Y(t_1, t_2)}{\partial t_1} = h_1^Y(t_1, t_2) \left( \frac{g'(t_1)}{g(t_1)} + h_1^Y(t_1, t_2) \right)$ .

**Theorem 3.2** *Let  $X$  and  $Y$  be two absolutely continuous random variables satisfying PRHRM with proportionality constant  $\theta(> 0)$ . A relationship of the form*

$$H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_1^Y(t_1, t_2) = (1 - \theta) [\mathcal{G}_Z^w(t_1, t_2) - m_X(t_1, t_2) \ln(t_1 - \alpha)], \quad (3.15)$$

where  $\mathcal{G}_Z^w(t_1, t_2) = E[X \ln(X - \alpha) | t_1 < X < t_2]$  and  $\alpha < t_1 < t_2 < \beta$ , holds if and only if  $X$  denotes the random lifetime of a component with uniform distribution over  $(\alpha, \beta)$ .

Proof: The *if part* is obtained from (2.8). To prove the converse, let us assume that (3.15) holds. Then from definition we can write

$$\begin{aligned} & - \int_{t_1}^{t_2} x f(x) \ln \frac{g(x)}{G(t_2) - G(t_1)} dx + \ln \frac{g(t_1)}{G(t_2) - G(t_1)} \int_{t_1}^{t_2} x f(x) dx \\ & = (1 - \theta) \left[ \int_{t_1}^{t_2} x \ln(x - \alpha) f(x) dx - \ln(t_1 - \alpha) \int_{t_1}^{t_2} x f(x) dx \right]. \end{aligned} \quad (3.16)$$

Differentiating (3.16) with respect to  $t_i$ ,  $i = 1, 2$  we get, after some algebraic calculations,

$$g(t_i) = k(t_i - \alpha)^{\theta-1}, \quad i = 1, 2 \text{ and } k > 0 \text{ (constant),}$$

or  $g(t) = k(t - \alpha)^{\theta-1}$ , which gives the required result.  $\square$

**Corollary 3.1** *Under PRHRM the relation*

$$H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_2^Y(t_1, t_2) = (1 - \theta) [\mathcal{G}_Z^w(t_1, t_2) - m_X(t_1, t_2) \ln(t_2 - \alpha)],$$

where  $\mathcal{G}_Z^w(t_1, t_2) = E[X \ln(X - \alpha) | t_1 < X < t_2]$  and  $\alpha < t_1 < t_2 < \beta$  characterizes the uniform distribution over  $(\alpha, \beta)$ .  $\square$

Next, we give a theorem which characterizes the power distribution.

**Theorem 3.3** *For two absolutely continuous random variables  $X$  and  $Y$  satisfying PRHRM with proportionality constant  $\theta(> 0)$ , the relation*

$$H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_1^Y(t_1, t_2) = (1 - c\theta) [\mathcal{G}_X^w(t_1, t_2) - m_X(t_1, t_2) \ln t_1], \quad (3.17)$$

for all  $0 < t_1 < t_2 < b$ , characterizes the power distribution

$$F(t) = \begin{cases} \left(\frac{t}{b}\right)^c, & 0 < t < b, \quad b, c > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (3.18)$$

Proof: If  $X$  follows power distribution as given in (3.18), then (3.17) is obtained from (2.8). To prove the converse, let us assume that (3.17) holds. Then differentiating with respect to  $t_i$ ,  $i = 1, 2$ , we get, after some algebraic calculations,

$$g(t_i) = kt_i^{c\theta-1}, \quad i = 1, 2 \text{ and } k > 0 \text{ (constant),}$$

or  $g(t) = kt^{c\theta-1}$ , which gives the required result.  $\square$

**Corollary 3.2** *The relationship*

$$H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_2^Y(t_1, t_2) = (1 - c\theta) [\mathcal{G}_X^w(t_1, t_2) - m_X(t_1, t_2) \ln t_2]$$

characterizes the power distribution as given in (3.18) under PRHRM.  $\square$

Below we characterize Weibull distribution under PHRM.

**Theorem 3.4** *Let  $X$  and  $Y$  be two absolutely continuous random variables satisfying PHRM with proportionality constant  $\theta(> 0)$ . A relationship of the form*

$$\begin{aligned} H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_1^Y(t_1, t_2) &= (1-p) [\mathcal{G}_X^w(t_1, t_2) - m_X(t_1, t_2) \ln t_1] \\ &+ \lambda \theta [m_{X^{p+1}}(t_1, t_2) - t_1^p m_X(t_1, t_2)], \end{aligned} \quad (3.19)$$

where  $m_{X^{p+1}}(t_1, t_2) = E(X^{p+1} | t_1 < X < t_2)$ , the conditional expectation of  $X^{p+1}$ , holds for all  $(t_1, t_2) \in D$  and  $p > 0$  if and only if  $X$  follows Weibull distribution

$$\overline{F}(t) = e^{-\lambda t^p}, \quad t > 0, \quad p > 0.$$

Proof: The *if part* is straight forward. To prove the converse, let us assume that (3.19) holds. Then differentiating with respect to  $t_i$ ,  $i = 1, 2$ , we get, after some algebraic calculations,

$$g(t_i) = k t_i^{p-1} e^{-\lambda \theta t_i^p}, \quad i = 1, 2 \text{ and } k > 0 \text{ (constant)},$$

or  $g(t) = k t^{p-1} e^{-\lambda \theta t^p}$ , which gives the required result.  $\square$

**Corollary 3.3** *Under PHRM, the relation*

$$\begin{aligned} H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_2^Y(t_1, t_2) &= (1-p) [\mathcal{G}_X^w(t_1, t_2) - m_X(t_1, t_2) \ln t_2] \\ &+ \lambda \theta [m_{X^{p+1}}(t_1, t_2) - t_2^p m_X(t_1, t_2)], \end{aligned}$$

where  $m_{X^{p+1}}(t_1, t_2) = E(X^{p+1} | t_1 < X < t_2)$ , the conditional expectation of  $X^{p+1}$ , characterizes the Weibull distribution as given in the above theorem.  $\square$

**Remark 3.1** *Taking  $p = 1$  in Theorem 3.4, we obtain the characterization theorem for exponential distribution with mean  $1/\lambda$ . Similarly,  $p = 2$  characterizes the Rayleigh distribution  $\overline{F}(t) = e^{-\lambda t^2}$ ,  $t > 0$ .  $\square$*

Now we consider Pareto-type distributions which are flexible parametric models and play important role in reliability, actuarial science, economics, finance and telecommunications. Arnold (1983) proposed a general version of this family of distributions called Pareto-IV distribution having the cumulative distribution function

$$F(x) = 1 - \left[ 1 + \left( \frac{x - \mu}{\beta} \right)^{\frac{1}{\gamma}} \right]^{-\alpha}, \quad x > \mu, \quad (3.20)$$

where  $-\infty < \mu < \infty$ ,  $\beta > 0$ ,  $\gamma > 0$  and  $\alpha > 0$ . This distribution is related to many other families of distributions. For example, setting  $\alpha = 1$ ,  $\gamma = 1$  and  $(\gamma = 1, \mu = \beta)$  in (3.20), one at a time, we obtain Pareto-III, Pareto-II and Pareto-I distributions, respectively. Also, taking  $\mu = 0$  and  $\gamma \rightarrow \frac{1}{\gamma}$  in (3.20), we obtain Burr-XII distribution.

Now we consider Pareto-type distributions for characterization under PHRM. Below we provide characterization of Pareto-I distribution.

**Theorem 3.5** *Let  $X$  and  $Y$  be two absolutely continuous random variables satisfying PHRM with proportionality constant  $\theta(> 0)$ . Then the relation*

$$H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_1^Y(t_1, t_2) = (\alpha\theta + 1) [\mathcal{G}_X^w(t_1, t_2) - m_X(t_1, t_2) \ln t_1], \quad (3.21)$$

*holds for all  $\beta < t_1 < t_2$  if and only if  $X$  follows Pareto-I distribution given by*

$$F(t) = 1 - \left(\frac{\beta}{t}\right)^\alpha, \quad t > \beta, \quad \alpha, \beta > 0.$$

Proof: The *if part* is straightforward. To prove the converse, let us assume that (3.21) holds. Then differentiating with respect to  $t_i$ ,  $i = 1, 2$ , we get, after some algebraic calculations,

$$g(t_i) = kt_i^{-(\alpha\theta+1)}, \quad i = 1, 2 \text{ and } k > 0 \text{ (constant),}$$

or  $g(t) = kt^{-(\alpha\theta+1)}$ , which gives the required result.  $\square$

**Corollary 3.4** *The relation*

$$H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_2^Y(t_1, t_2) = (\alpha\theta + 1) [\mathcal{G}_X^w(t_1, t_2) - m_X(t_1, t_2) \ln t_2]$$

*characterizes the same distribution under PHRM as mentioned in the above theorem.*  $\square$

We conclude this section by characterizing Pareto-II distribution. The proof is similar to that of Theorem 3.5 and hence omitted.

**Theorem 3.6** *Let  $X$  and  $Y$  be two absolutely continuous random variables satisfying PHRM with proportionality constant  $\theta(> 0)$ . Then the relation*

$$H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_1^Y(t_1, t_2) = (\alpha\theta + 1) [\mathcal{G}_Z^w(t_1, t_2) - m_X(t_1, t_2) \ln(t_1 - \mu + \beta)], \quad (3.22)$$

*where  $\mathcal{G}_Z^w(t_1, t_2) = E[X \ln(X - \mu + \beta) | t_1 < X < t_2]$  holds for all  $\mu < t_1 < t_2$  if and only if  $X$  follows Pareto-II distribution given by*

$$F(t) = 1 - \left[1 + \left(\frac{t - \mu}{\beta}\right)\right]^{-\alpha}, \quad t > \mu.$$

**Corollary 3.5** *Under PHRM the relation*

$$H_{X,Y}^w(t_1, t_2) + m_X(t_1, t_2) \ln h_2^Y(t_1, t_2) = (\alpha\theta + 1) [\mathcal{G}_Z^w(t_1, t_2) - m_X(t_1, t_2) \ln(t_2 - \mu + \beta)],$$

*where  $\mathcal{G}_Z^w(t_1, t_2) = E[X \ln(X - \mu + \beta) | t_1 < X < t_2]$  and  $\mu < t_1 < t_2$  characterizes the same distribution as mentioned in the above theorem.*

## 4 Monotonic transformations

In this section we study the weighted interval inaccuracy measure under strict monotonic transformations. The following result is a generalization of Theorem 4.1 of Di Crescenzo and Longobardi (2006).

**Theorem 4.1** *Let  $X$  and  $Y$  be two absolutely continuous nonnegative random variables. Suppose  $\varphi(x)$  is strictly monotonic, continuous and differentiable function with derivative  $\varphi'(x)$ . Then, for all  $0 < t_1 < t_2 < \infty$ ,*

$$H_{\varphi(X), \varphi(Y)}^w(t_1, t_2) = \begin{cases} H_{X,Y}^{w,\varphi}(\varphi^{-1}(t_1), \varphi^{-1}(t_2)) \\ + E[\varphi(X) \ln \varphi'(X) | \varphi^{-1}(t_1) < X < \varphi^{-1}(t_2)], \text{ } \varphi \text{ strictly increasing} \\ H_{X,Y}^{w,\varphi}(\varphi^{-1}(t_2), \varphi^{-1}(t_1)) \\ + E[\varphi(X) \ln \{-\varphi'(X)\} | \varphi^{-1}(t_2) < X < \varphi^{-1}(t_1)], \text{ } \varphi \text{ strictly decreasing,} \end{cases}$$

where

$$H_{X,Y}^{w,\varphi}(t_1, t_2) = - \int_{t_1}^{t_2} \varphi(x) \frac{f(x)}{F(t_2) - F(t_1)} \ln \frac{g(x)}{G(t_2) - G(t_1)} dx.$$

Proof: Let  $\varphi(x)$  be strictly increasing. Then from (1.4), (1.5) and (1.6) we have

$$H_{\varphi(X), \varphi(Y)}^w = H_{X,Y}^{w,\varphi} + E[\varphi(X) \ln \varphi'(X)], \quad (4.23)$$

where  $H_{X,Y}^{w,\varphi} = - \int_0^\infty \varphi(x) f(x) \ln g(x) dx$ ,

$$H_{\varphi(X), \varphi(Y)}^w(t) = H_{X,Y}^{w,\varphi}(\varphi^{-1}(t)) + E[\varphi(X) \ln \varphi'(X) | X > \varphi^{-1}(t)], \quad (4.24)$$

where  $H_{X,Y}^{w,\varphi}(t) = - \int_t^\infty \varphi(x) \frac{f(x)}{F(t)} \ln \left( \frac{g(x)}{G(t)} \right) dx$ , and

$$\overline{H}_{\varphi(X), \varphi(Y)}^w(t) = \overline{H}_{X,Y}^{w,\varphi}(\varphi^{-1}(t)) + E[\varphi(X) \ln \varphi'(X) | X < \varphi^{-1}(t)], \quad (4.25)$$

where  $\overline{H}_{X,Y}^{w,\varphi}(t) = - \int_0^t \varphi(x) \frac{f(x)}{F(t)} \ln \left( \frac{g(x)}{G(t)} \right) dx$ . Now from Remark 2.3 we can write

$$\begin{aligned} H_{\varphi(X), \varphi(Y)}^w &= F(\varphi^{-1}(t_1)) \overline{H}_{\varphi(X), \varphi(Y)}^w(t_1) + [F(\varphi^{-1}(t_2)) - F(\varphi^{-1}(t_1))] H_{\varphi(X), \varphi(Y)}^w(t_1, t_2) \\ &\quad + \overline{F}(\varphi^{-1}(t_2)) H_{\varphi(X), \varphi(Y)}^w(t_2) - E(\varphi(X)) [F^{w,\varphi}(\varphi^{-1}(t_1)) \ln G(\varphi^{-1}(t_1)) \\ &\quad + \{F^{w,\varphi}(\varphi^{-1}(t_2)) - F^{w,\varphi}(\varphi^{-1}(t_1))\} \ln \{G(\varphi^{-1}(t_2)) - G(\varphi^{-1}(t_1))\} \\ &\quad + \overline{F}^{w,\varphi}(\varphi^{-1}(t_2)) \ln \overline{G}(\varphi^{-1}(t_2))], \end{aligned}$$

where  $F^{w,\varphi}(t) = \frac{1}{E[\varphi(X)]} \int_0^t \varphi(x) f(x) dx$ . On using (4.23), (4.24) and (4.25) we obtain

$$\begin{aligned} H_{X,Y}^{w,\varphi} + E[\varphi(X) \ln \varphi'(X)] &= [F(\varphi^{-1}(t_2)) - F(\varphi^{-1}(t_1))] H_{\varphi(X), \varphi(Y)}^w(t_1, t_2) \\ &\quad + F(\varphi^{-1}(t_1)) E[\varphi(X) \ln \varphi'(X) | X < \varphi^{-1}(t_1)] + \overline{F}(\varphi^{-1}(t_2)) E[\varphi(X) \ln \varphi'(X) | X > \varphi^{-1}(t_2)] \\ &\quad + F(\varphi^{-1}(t_1)) \overline{H}_{X,Y}^{w,\varphi}(\varphi^{-1}(t_1)) + \overline{F}(\varphi^{-1}(t_2)) H_{X,Y}^{w,\varphi}(\varphi^{-1}(t_2)) \\ &\quad - E(\varphi(X)) [F^{w,\varphi}(\varphi^{-1}(t_1)) \ln G(\varphi^{-1}(t_1)) + \overline{F}^{w,\varphi}(\varphi^{-1}(t_2)) \ln \overline{G}(\varphi^{-1}(t_2)) \\ &\quad + \{F^{w,\varphi}(\varphi^{-1}(t_2)) - F^{w,\varphi}(\varphi^{-1}(t_1))\} \ln \{G(\varphi^{-1}(t_2)) - G(\varphi^{-1}(t_1))\}], \end{aligned} \quad (4.26)$$

where the last three terms on the right hand side of (4.26) are equal to

$$H_{X,Y}^{w,\varphi} - [F(\varphi^{-1}(t_2)) - F(\varphi^{-1}(t_1))] H_{X,Y}^{w,\varphi}(\varphi^{-1}(t_1), \varphi^{-1}(t_2)),$$

giving the first part of the proof. If  $\varphi(x)$  is strictly decreasing we similarly obtain the second part of the proof.  $\square$

**Remark 4.1** Let  $\varphi_1(x) = F(x)$  and  $\varphi_2(x) = \overline{F}(x)$ , with  $\varphi_1$  and  $\varphi_2$  satisfying the assumptions of Theorem 4.1. Here  $\varphi_1(X)$  and  $\varphi_2(X)$  are uniformly distributed over  $(0, 1)$ . Then, for all  $(t_1, t_2) \in D$ , we have

$$H_{F(X), F(Y)}^w(t_1, t_2) = H_{X,Y}^{w,F}(F^{-1}(t_1), F^{-1}(t_2)) + E[F(X) \ln f(X) | F^{-1}(t_1) < X < F^{-1}(t_2)]$$

and

$$H_{\overline{F}(X), \overline{F}(Y)}^w(t_1, t_2) = H_{X,Y}^{w,\overline{F}}(\overline{F}^{-1}(t_2), \overline{F}^{-1}(t_1)) + E[\overline{F}(X) \ln f(X) | \overline{F}^{-1}(t_2) < X < \overline{F}^{-1}(t_1)].$$

**Remark 4.2** For two absolutely continuous nonnegative random variables  $X$  and  $Y$

$$H_{aX, aY}^w(t_1, t_2) = aH_{X,Y}^w\left(\frac{t_1}{a}, \frac{t_2}{a}\right) + m_X\left(\frac{t_1}{a}, \frac{t_2}{a}\right) a \ln a$$

for all  $a > 0$  and  $t_1 > 0$ . Furthermore, for all  $0 < b < t_1$

$$H_{X+b, Y+b}^w(t_1, t_2) = H_{X,Y}^w(t_1 - b, t_2 - b) + bH_{X,Y}(t_1 - b, t_2 - b).$$

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